

# On the asymptotic expansion of $\Gamma(x)$ , Lagrange's inversion theorem and the Stirling coefficients

R. B. PARIS

*University of Abertay Dundee, Dundee DD1 1HG, UK*  
E-Mail: r.paris@abertay.ac.uk

## Abstract

We show how the asymptotic expansion for the gamma function  $\Gamma(x)$ , similar to that obtained by Boyd [Proc. Roy. Soc. London **A447** (1994) 609–630], can be obtained by using a form of Lagrange's inversion theorem with a remainder. A (possibly) new closed-form representation for the Stirling coefficients is given.

**Mathematics Subject Classification:** 33B15, 34E05, 30E15, 41A60

**Keywords:** Gamma function, asymptotic expansion, Lagrange's inversion theorem, representation for the Stirling coefficients

## 1. Introduction

The gamma function  $\Gamma(x)$  has the well-known asymptotic expansion as  $x \rightarrow \infty$

$$\Gamma(x) = \int_0^\infty e^{-\tau} \tau^{x-1} d\tau \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \sum_{n=0}^{\infty} \frac{(-)^n \gamma_n}{x^n}, \quad (1.1)$$

where  $\gamma_n$  are the so-called Stirling coefficients, the first few being (with  $\gamma_0 = 1$ )

$$\gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}, \quad \gamma_4 = -\frac{571}{2488320}.$$

The above expansion holds for large complex  $x$  in the sector  $|\arg x| \leq \pi - \delta$ ,  $\delta > 0$ , although in this note we shall restrict our attention throughout to positive values of  $x$ . The slowly varying part of  $\Gamma(x)$  (when  $x$  is large) is given by

$$\Gamma^*(x) = \frac{\Gamma(x)}{\sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}} \quad (1.2)$$

and, from (1.1), its asymptotic expansion is

$$\Gamma^*(x) \sim \sum_{n=0}^{\infty} \frac{(-)^n \gamma_n}{x^n} = 1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \cdots \quad (x \rightarrow \infty).$$

Employing the reformulation of the method of steepest descents developed by Berry & Howls [2] (for a summary, see [10, pp. 94–99]), Boyd [3] established the result for positive integer  $m$

$$\Gamma^*(x) = \sum_{n=0}^{m-1} \frac{(-)^n \gamma_n}{x^n} + \tilde{R}_m(x), \quad (1.3)$$

where

$$\tilde{R}_m(x) = \frac{x^{-m}}{\sqrt{2\pi}} \int_0^\infty e^{-w} w^{m-\frac{1}{2}} \frac{1}{2\pi i} \int_{C'} \frac{\{h(z)\}^{-m+\frac{1}{2}}}{h(z) - w/x} dz dw. \quad (1.4)$$

The quantity  $h(z) = e^z - 1 - z$  and  $C'$  (for  $m \geq 1$ ) is a contour that can be taken to be a pair of straight parallel lines situated on either side of the real  $z$ -axis. By expanding the contour  $C'$  to coincide with the other saddle points of the integrand in (1.1), Boyd then obtained the elegant expression

$$\tilde{R}_m(x) = \frac{i^m x^{-m}}{2\pi i} \int_0^\infty s^{m-1} e^{-2\pi s} \left\{ \frac{\Gamma^*(is)}{1 - is/x} - (-)^m \frac{\Gamma^*(-is)}{1 + is/x} \right\} ds,$$

from which he was able to derive a bound on  $\tilde{R}_m(x)$  (valid for complex  $x$ ). This bound has been recently improved in [8] by employing more refined bounds on  $\Gamma^*(is)$ .

The Stirling coefficients appearing in the expansions (1.1) and (1.3) can be generated numerically by means of the following recurrence relation:

$$\gamma_n = (-2)^n \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}} d_{2n},$$

$$d_n = \frac{n+1}{n+2} \left\{ \frac{d_{n-1}}{n} - \sum_{j=1}^{n-1} \frac{d_j d_{n-j}}{j+1} \right\} \quad (n \geq 1),$$

where  $d_0 = 1$  and an empty sum is interpreted as zero. A closed-form representation involving the 3-associated Stirling number  $S_3(\ell, k)$  is found in [5] as

$$\gamma_n = \sum_{j=0}^{2n} \frac{(-)^j S_3(2j + 2n, j)}{2^{j+n} (j+n)!},$$

where

$$\exp \left[ u \left( \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \right) \right] = \sum_{k, \ell \geq 0} S_3(\ell, k) \frac{u^k t^\ell}{\ell!}.$$

A proof of this result is given in [2]. A different representation has been obtained recently in [7] in the form

$$\gamma_n = \sum_{m=0}^{2n} \sum_{r=0}^m \frac{(\frac{1}{2})_{m+n}}{r! 2^{r-m-n}} \sum_{j=0}^{m-r} \frac{(-)^{j+n} S_{2m+2n-2r-j}^{(m-r-j)}}{j! (2m+2n-2r-j)!},$$

where  $S_k^{(m)}$  denotes the Stirling number of the first kind [1, p. 824].

In this note we obtain the expansion of  $\Gamma^*(x)$  in the form (1.3) and (1.4) by making use of Lagrange's inversion theorem with a remainder, so that the inversion is valid on an infinite interval. The derivation of the remainder in Lagrange's inversion theorem is given in the appendix. The approach we use also provides a (possibly) new closed-form representation for the Stirling coefficients.

## 2. The expansion for $\Gamma^*(x)$ as $x \rightarrow \infty$

We make the change of variable  $t = \log(\tau/x)$  in Euler's integral representation for  $\Gamma(x)$  in (1.1) to find

$$\Gamma(x) = x^x e^{-x} \int_{-\infty}^{\infty} e^{-xh(t)} dt,$$

where

$$h(t) = e^t - t - 1.$$

The scaled gamma function defined in (1.2) then becomes

$$\Gamma^*(x) = \frac{x^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-xh(t)} dt. \quad (2.1)$$

The function  $h(t)$  has saddle points (where  $h'(t) = 0$ ) at  $t = 2\pi ki$ ,  $k = 0, \pm 1, \pm 2, \dots$ . The saddle at  $t = 0$  is the active saddle and the integration path in (2.1) coincides with the paths of steepest descent from the origin. We now make the quadratic transformation

$$h(t) = \frac{1}{2}u^2 \quad (2.2)$$

with the assumption that  $\text{sign}(t) = \text{sign}(u)$ , to yield

$$\Gamma^*(x) = \frac{x^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}xu^2} \frac{dt}{du} du. \quad (2.3)$$

To proceed we require the inversion of (2.2) to express  $t$  as a function of the new variable  $u$ . Following the argument given in [6, p. 54], it can be seen that the inversion  $t(u)$  is a many-valued function with branch points at  $u = 0$  and  $u = \pm 2\sqrt{\pi k}e^{\pm \pi i/4}$ ,  $k = 1, 2, \dots$ . Since

$$\frac{dt}{du} = \frac{u}{e^t - 1},$$

the only singularities of  $t(u)$  are at these branch points, and so the series expansion of  $t(u)$  will converge in  $|u| < 2\sqrt{\pi}$ .

### 2.1 The derivation of the expansion for $\Gamma^*(x)$

We employ Lagrange's inversion theorem with a remainder given in the appendix to obtain the inversion  $t(u)$  valid for  $u \in [0, \infty)$ . Writing (2.2) in the form

$$u = \frac{t}{\phi(t)}, \quad \phi(t) = \left( \frac{\frac{1}{2}t^2}{e^t - t - 1} \right)^{1/2} = \left( 1 + 2 \sum_{r=1}^{\infty} \frac{t^r}{(r+2)!} \right)^{-1/2}, \quad (2.4)$$

we have from (A.4)

$$t = \sum_{n=1}^{m-1} \frac{u^n}{n!} D^{n-1} \phi^n(0) - \frac{u^m}{(m-1)!} D^{m-1} \phi^m(0) + Q_m(u)$$

for positive integer  $m$ , where  $D^k \phi(0) \equiv (d/dt)^k \phi(t)|_{t=0}$  ( $k = 0, 1, 2, \dots$ ), and

$$Q_m(u) = \frac{u^m}{2\pi i} \oint_C \frac{1 - u\phi'(z)}{z - u\phi(z)} \frac{\phi^m(z)}{z^{m-1}} dz. \quad (2.5)$$

The contour  $C$  denotes a closed path described in the positive sense surrounding the points  $z = 0$  and  $z = t$ . Making the change of summation index  $m \rightarrow 2m$  and differentiating we find

$$\frac{dt}{du} = \sum_{n=0}^{m-1} \frac{u^{2n}}{(2n)!} D^{2n} \phi^{2n+1}(0) + \text{odd terms in } u + \frac{d}{du} Q_{2m}(u), \quad (2.6)$$

where we have not specified the terms in the finite sum with odd parity in  $u$  since they make no contribution to the integral in (2.3).

Substitution of the expansion (2.6) into (2.3) then produces

$$\begin{aligned}\Gamma^*(x) &= \frac{x^{\frac{1}{2}}}{\sqrt{2\pi}} \sum_{n=0}^{m-1} \frac{D^{2n} \phi^{2n+1}(0)}{(2n)!} \int_{-\infty}^{\infty} u^{2n} e^{-\frac{1}{2}xu^2} du + R_m(x) \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{m-1} \frac{2^n \Gamma(n + \frac{1}{2})}{(2n)! x^n} D^{2n} \phi^{2n+1}(0) + R_m(x),\end{aligned}\tag{2.7}$$

where the remainder after  $m$  terms  $R_m(x)$  is given by

$$R_m(x) = \frac{x^{\frac{3}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-\frac{1}{2}xu^2} Q_{2m}(u) du.\tag{2.8}$$

Identification of the coefficients in the finite sum in terms of the Stirling coefficients  $\gamma_n$  (see (1.1)) then yields

$$\Gamma^*(x) = \sum_{n=0}^{m-1} \frac{(-)^n \gamma_n}{x^n} + R_m(x),\tag{2.9}$$

where

$$\gamma_n = \frac{(-2)^n}{\sqrt{\pi}} \frac{\Gamma(n + \frac{1}{2})}{(2n)!} D^{2n} \phi^{2n+1}(0) = \frac{(-)^n}{2^n n!} D^{2n} \phi^{2n+1}(0).\tag{2.10}$$

## 2.2 An integral representation for the remainder $R_m(x)$

Substituting the representation of  $Q_m(u)$  in (2.5) into the expression for the remainder  $R_m(x)$  in (2.8) we obtain

$$\begin{aligned}R_m(x) &= \frac{x^{\frac{3}{2}}}{\sqrt{2\pi}} \int_0^{\infty} u^{2m+1} e^{-\frac{1}{2}xu^2} \frac{1}{2\pi i} \oint_C \frac{\phi^{2m}(z)}{z^{2m-1}} \left\{ \frac{1 - u\phi'(z)}{z - u\phi(z)} - \frac{1 + u\phi'(z)}{z + u\phi(z)} \right\} dz du \\ &= \frac{2x^{\frac{3}{2}}}{\sqrt{2\pi}} \int_0^{\infty} u^{2m+2} e^{-\frac{1}{2}xu^2} \frac{1}{2\pi i} \oint_C \frac{z(-\phi(z)/z)'}{1 - u^2(\phi(z)/z)^2} (\phi(z)/z)^{2m} dz du.\end{aligned}$$

Since, from (2.4),

$$\phi(z)/z = (2h(z))^{-1/2}, \quad (\phi(z)/z)' = -h'(z)/(2^{3/2}h^{3/2}(z)),$$

we then find after some straightforward rearrangement, together with the change of variable  $w = \frac{1}{2}xu^2$ , that

$$R_m(x) = \frac{x^{-m}}{\sqrt{2\pi}} \int_0^{\infty} e^{-w} w^{m+\frac{1}{2}} \frac{1}{2\pi i} \oint_C \frac{zh'(z)}{h(z) - w/x} \{h(z)\}^{-m-\frac{1}{2}} dz dw,\tag{2.11}$$

where the contour  $C$  denotes a closed path described in the positive sense surrounding the points  $z = 0$  and the two zeros (one positive and one negative) of  $h(z) = w/x$ .

**Remark 1.** As in (1.4), the contour  $C$  in (2.11) can be replaced by  $C'$  which is a pair of parallel lines just above and below the real  $z$ -axis.

**Remark 2.** Referring to (1.4), we see that Boyd's expression for the remainder after  $m$  terms is given by

$$\frac{x^{-m}}{\sqrt{2\pi}} \int_0^\infty e^{-w} w^{m-\frac{1}{2}} \frac{1}{2\pi i} \oint_{C'} \frac{\{h(z)\}^{-m+\frac{1}{2}}}{h(z) - w/x} dz du. \quad (2.12)$$

We have been unable to demonstrate the equivalence between this form of the remainder and that in (2.11). We believe, however, that these two expressions are equivalent, a conjecture that is supported by high-precision numerical evaluation of the double integrals using *Mathematica*. In the particular case  $m = 2$ ,  $x = 8$  for example, we found agreement between the remainder terms in (2.11) and (2.12) to more than 30dp.

### 3. A representation for the Stirling coefficients $\gamma_n$

Our representation for the Stirling coefficients is given in the following theorem.

**Theorem 1.** *The Stirling coefficients  $\gamma_n$  ( $n \geq 1$ ) are given by*

$$\gamma_n = 2^n \sum \frac{(-2)^m (\frac{1}{2})_{m+n}}{\prod_{k=1}^{2n} m_k! ((k+2)!)^{m_k}}, \quad (3.1)$$

where  $(a)_n = \Gamma(a+n)/\Gamma(a)$  is Pochhammer's symbol,

$$m = m_1 + m_2 + \cdots + m_{2n}$$

and the summation is taken over all nonnegative integer solutions  $(m_1, \dots, m_{2n})$  of the partition

$$P_{2n} = \{(m_1, m_2, \dots, m_{2n}) : \sum_{k=1}^{2n} k m_k = 2n\}. \quad (3.2)$$

*Proof.* From (2.10), the Stirling coefficients  $\gamma_n$  are given by

$$\gamma_n = \frac{(-2)^n}{\sqrt{\pi}} \frac{\Gamma(n + \frac{1}{2})}{(2n)!} D^{2n} \phi^{2n+1}(0), \quad (3.3)$$

where  $\phi(t)$  is defined in (2.4). To evaluate the derivatives  $D^{2n} \phi^{2n+1}(0)$  we make use of Faà di Bruno's formula [1, p. 823], [9, p. 5]

$$\frac{d^n}{dt^n} [f(g(t))] = n! \sum f^{(m)}(g(t)) \prod_{k=1}^n \frac{1}{m_k!} \left( \frac{g^{(k)}(t)}{k!} \right)^{m_k}, \quad (3.4)$$

where

$$m = m_1 + m_2 + \cdots + m_n$$

and the summation is taken over all nonnegative integer solutions  $(m_1, \dots, m_n)$  of the partition

$$m_1 + 2m_2 + \cdots + n m_n = n.$$

From (2.4), we set  $f(u) = u^{-n-1/2}$  and  $g(t) = 1 + 2 \sum_{r=1}^\infty t^r / (r+2)!$ . Then a simple calculation shows that

$$f^{(k)}(1) = (-)^k \frac{\Gamma(n + k + \frac{1}{2})}{\Gamma(n + \frac{1}{2})}, \quad g^{(k)}(0) = \frac{2}{(k+1)(k+2)}$$

for  $k = 1, 2, \dots$ . From (3.4) we then obtain

$$D^{2n} \phi^{2n+1}(0) = \frac{(2n)! \sqrt{\pi}}{\Gamma(n + \frac{1}{2})} \sum \frac{(-2)^m (\frac{1}{2})_{m+n}}{\prod_{k=1}^{2n} m_k! ((k+2)!)^{m_k}}.$$

Substitution of these values into (3.3) then yields the result in (3.1).  $\square$

An alternative version of (3.1) is

$$\gamma_n = \frac{2^n}{(2n)!} \sum \frac{(-2)^m (\frac{1}{2})_{m+n} C_{\vec{m}}}{\prod_{k=1}^{2n} ((k+1)(k+2))^{m_k}}, \quad (3.5)$$

where the coefficients  $C_{\vec{m}}$  are given by<sup>1</sup>

$$C_{\vec{m}} = \prod_{k=1}^{2n} \frac{(2n)!}{m_k! (k!)^{m_k}}.$$

Values of these coefficients for  $n \leq 5$  are tabulated in [1, p. 831].

#### 4. Concluding remarks

In (2.7) we have obtained the expansion of the scaled gamma function  $\Gamma^*(x)$  as a finite sum involving inverse powers of  $x$  together with a remainder  $R_m(x)$  using Lagrange's inversion theorem. This result is similar to that found by Boyd [3] who employed the Berry-Howls reformulation of the treatment of Laplace-type integrals. From this we derived an expression for the Stirling coefficients  $\gamma_n$  given in (2.10) and in Theorem 1.

A superficially similar procedure (but not equivalent) has been described by Brassesco and Méndez [4]. They started with the result<sup>2</sup> (for  $x > 0$ )

$$\Gamma(x) = x^x \int_0^\infty e^{-xt} t^x dt$$

and made the *linear* transformation  $t \rightarrow 1 + w$ ,  $w = ux^{-1/2}$  to obtain

$$\begin{aligned} \Gamma(x) &= x^x e^{-x} \int_{-1}^\infty e^{x\{\log(1+w)-w\}} dw \\ &= x^{x-\frac{1}{2}} e^{-x} \int_{-\sqrt{x}}^\infty e^{-u^2/2} e^{u^2 \lambda(w)} du, \end{aligned}$$

where

$$\lambda(z) = z^{-2} \{\log(1+z) - z + \frac{1}{2}z^2\}.$$

Substituting the Maclaurin expansion

$$e^{u^2 \lambda(z)} = \sum_{j \geq 0} \frac{z^j}{j!} D^j e^{u^2 \lambda(z)}|_{z=0}, \quad D \equiv \frac{d}{dz} \quad (4.1)$$

<sup>1</sup>In [1, p. 831] these quantities are called  $M_3 = (2n; m_1, m_2, \dots, m_{2n})'$ .

<sup>2</sup>This follows from the Euler integral for  $\Gamma(x+1)$ , followed by the change of variable  $\tau \rightarrow xt$  and use of the result  $\Gamma(x+1) = x\Gamma(x)$ .

with  $z$  replaced by  $ux^{-1/2}$  into the above integral, they found upon reversal of the order of summation and integration

$$\Gamma(x) \sim x^{x-\frac{1}{2}} e^{-x} \sum_{j \geq 0} \frac{x^{-j/2}}{j!} D^j \left( \int_{-\sqrt{x}}^{\infty} u^j e^{-u^2 \Lambda(z)/2} du \right)_{z=0}, \quad \Lambda(z) = 1 - 2\lambda(z). \quad (4.2)$$

The above integral is then extended over  $(-\infty, \infty)$ , so that the terms with odd index  $j$  vanish, to yield the representation for the Stirling coefficients

$$\gamma_n = \frac{(-1)^n}{2^n n!} D^{2n} \Lambda^{-n-\frac{1}{2}}(0). \quad (4.3)$$

This representation is equivalent to that in (2.10).

The implication here is that the evaluation of the  $\gamma_n$  by this means has resulted in the neglect of exponentially small terms produced by extending the above integral to include the interval  $(-\infty, -\sqrt{x})$ . In addition, Brassesco and Méndez [4, Eq. (2.26)] incorrectly write (4.2) as an equality when this cannot be the case since the expansion (4.1) is convergent in  $|z| < 1$ . This fact results in integration of the series on  $[0, \infty)$  beyond its interval of convergence. In our treatment, we make the quadratic transformation in (2.2) to obtain the Stirling coefficients expressed *exactly* in terms of an integral over the interval  $(-\infty, \infty)$ . This results in *no exponentially small terms being neglected*. Also the use of the Lagrange inversion theorem with a remainder circumvents the problem of integration beyond the interval of convergence (which in the case of  $t(u)$  in (2.4) is  $|u| < 2\sqrt{\pi}$ ) and leads to an expression for the remainder term in the expansion.

The closed-form expression for the Stirling coefficients  $\gamma_n$  in (3.1), and its alternative form (3.5), involves the partition  $P_{2n}$ . The cardinality of this set is equal to the partition function  $p(n)$ , where  $p(n)$  represents the number of partitions of the positive integer  $n$ . To illustrate the use of (3.5) we take the case  $n = 2$ , so that  $p(4) = 5$  and [1, p. 831]

$$P_4 = \{(0, 0, 0, 1), (1, 0, 1, 0), (0, 2, 0, 0), (2, 1, 0, 0), (4, 0, 0, 0)\},$$

$$C_{\vec{m}} = \{1, 4, 3, 6, 1\}.$$

Then

$$\gamma_2 = \frac{2^2}{4!} \left\{ -\frac{2 \cdot 1(\frac{1}{2})_3}{5 \cdot 6} + 4(\frac{1}{2})_4 \left( \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{3}{(3 \cdot 4)^2} \right) - \frac{8 \cdot 6(\frac{1}{2})_5}{(2 \cdot 3)^2 3 \cdot 4} + \frac{16 \cdot 1(\frac{1}{2})_6}{(2 \cdot 3)^4} \right\} = \frac{1}{288}.$$

It is clear that  $p(n)$  grows rapidly with  $n$ . Consequently, (3.5) is not a practical means for the computation of these coefficients for large values of  $n$ .

### Appendix: The Lagrange expansion theorem with a remainder

Let  $f(t)$  and  $\phi(t)$  be analytic on and inside a simple closed contour  $C$  in the complex  $t$ -plane surrounding the point  $t = a$ . Suppose further that the function  $\psi(z) = z - a - u\phi(z)$  has only one root  $z = t$  inside  $C$  given by

$$t - a = u\phi(t), \quad (A.1)$$

where  $u$  is the expansion variable. The procedure we adopt is a modification of that presented in [11, p. 17].

Our starting point is the identity

$$f(t) = \frac{1}{2\pi i} \oint_C f(z) \frac{\psi'(z)}{\psi(z)} dz = \frac{1}{2\pi i} \oint_C f(z) \frac{1 - u\phi'(z)}{z - a - u\phi(z)} dz.$$

Upon expansion of the factor  $(z - a - u\phi(z))^{-1}$  as a finite geometric progression of  $m$  terms with a remainder, we find

$$f(t) = \frac{1}{2\pi i} \oint_C f(z)(1 - u\phi'(z)) \left\{ \sum_{n=0}^{m-1} \frac{u^n \phi^n(z)}{(z-a)^{n+1}} + \frac{u^m \phi^m(z)}{(z-a)^m(z-a-u\phi(z))} \right\} dz.$$

Making use of the Cauchy formula

$$F^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{F(\zeta)}{(\zeta - z)^{n+1}} d\zeta,$$

we obtain

$$\begin{aligned} f(t) &= \sum_{n=0}^{m-1} \frac{u^n}{n!} D^n [f(a)\phi^n(a)(1 - u\phi'(a))] + Q_m(u) \\ &= \sum_{n=0}^{m-1} \frac{u^n}{n!} D^n [f(a)\phi^n(a) - \frac{u}{n+1} f(a) D\phi^{n+1}(a)] + Q_m(u), \end{aligned}$$

where  $D \equiv d/da$ , the remainder  $Q_m(u)$  is given by

$$Q_m(u) = \frac{u^m}{2\pi i} \oint_C f(z) \frac{1 - u\phi'(z)}{z - a - u\phi(z)} \frac{\phi^m(z)}{(z-a)^m} dz, \quad (\text{A.2})$$

and the points  $z = a$  and  $z = t$  are enclosed by the contour  $C$ .

Straightforward rearrangement of the sum over  $n$  then yields Lagrange's expansion with a remainder in the form<sup>3</sup>

$$f(t) = f(a) + \sum_{n=1}^{m-1} \frac{u^n}{n!} D^{n-1} [f'(a)\phi^n(a)] - \frac{u^m}{m!} D^{m-1} [f(a) D\phi^m(a)] + Q_m(u) \quad (\text{A.3})$$

for positive integer  $m$ , where  $\phi(t)$  is specified by (A.1).

In the special case  $f(t) = t$  and  $a = 0$ , we have from (A.2) and (A.3) the expansion for positive integer  $m$

$$t = \sum_{n=1}^{m-1} \frac{u^n}{n!} D^{n-1} \phi^n(0) - \frac{u^m}{(m-1)!} D^{m-1} \phi^m(0) + \frac{u^m}{2\pi i} \oint_C \frac{1 - u\phi'(z)}{z - u\phi(z)} \frac{\phi^m(z)}{z^{m-1}} dz, \quad (\text{A.4})$$

where  $\phi(t)$  is specified in (A.1) and we have used the fact that

$$D^{m-1} [a D\phi^m(a)]_{a=0} = (m-1) D^{m-1} \phi^m(0)$$

and the contour  $C$  encloses the poles at  $z = 0$  and  $z = t$ .

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<sup>3</sup>We note that the usual form of this theorem [11, p. 17], [12, p. 133] has the additional requirement  $|u\phi(z)| < |z - a|$  for points on  $C$ , so that the arbitrary function  $f(t)$  then has the expansion

$$f(t) = f(a) + \sum_{n=1}^{\infty} \frac{u^n}{n!} D^{n-1} [f'(a)\phi^n(a)].$$



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